Stochastic Processes and their Applications 17 (1984) 369–373
North-Holland

SHORT COMMUNICATION

THE OCCURRENCE OF SEQUENCE PATTERNS IN ERGODIC MARKOV CHAINS

Rodolfo V. BENEVENTO
Dipartimento di Matematica dell'Università della Calabria, 87036 – Arcavacata di Rende, Italy

Received 28 June 1982
Revised 22 July 1983

The occupation measure identity is used to derive the expected waiting time for the first occurrence of a fixed finite pattern in a sequence of observations generated by an ergodic Markov chain.

hitting times * occupation measure * sequence patterns

1. Introduction

In [5] Shuo-Yen R. Li has given a martingale argument to derive the expectation of the number of i.i.d. experiments required to observe a given sequence of outcomes for the first time. In this way he has generalized some results on the duration of a search for a fixed pattern in random data. Further developments under the i.i.d. hypothesis can be found in the same paper as well as in [1], [2] and [3]. Another recent result in this line is in [4].

Here the i.i.d. hypothesis is replaced by the weaker assumption that the experiments form an ergodic Markov chain. Then Li's result [5, Lemma 2.4], with or without an initial state prescribed, follows from Theorem 3.1 below. The similarity of this theorem and Li's is particularly evident under the assumption of Corollary 3.2 that the chain starts at the last state of the sequence.

The proof makes use of the occupation measure identity for Markov chains [6, Theorem 3.5].

2. Number of repetitions of the first state of a given pattern before the entire pattern occurs

Let J be a countable set of states, let Ω be the space of all sequences of states and let $\mathscr{F}$ be the usual product σ-field generated by the sequence $X = \{X_n;\ n \geq 0\}$ of the coordinate maps $X_n : \Omega \to J$.

Let $P=\{P(i,j); i,j\in J\}$ be an ergodic transition matrix and, for each state i, let P_i be the probability on $(\Omega,\mathscr{F})$ which makes X a Markov chain with initial point i and transition matrix P.

Given a finite sequence of states $S=(j_0,\ldots,j_m)$ such that $P(j_{k-1},j_k)>0$ for $0<k\leqslant m$, let F be the event $(X_0=j_0,\ldots,X_m=j_m)$ and let $\theta_n:\Omega\to\Omega$ be the n-fold shift map.

With this notation and with 1_F standing for the indicator function we have

$$E_j 1_F=\begin{cases}P(j_0,j_1)\cdots P(j_{m-1},j_m) & \text{if } j=j_0,\\ 0 & \text{otherwise.}\end{cases} \tag{1}$$

We also have

$$1_F\circ\theta_n=1_{(X_n=j_0,\ldots,X_{n+m}=j_m)}.$$

The first-occurrence time of S is the r.v.

$$T=\min\{k\geqslant m;\ 1_F\circ\theta_{k-m}=1\}$$

and it is a finite stopping time for the chain X. This definition yields

$$\sum_{n=0}^{T-1} 1_F\circ\theta_n=\sum_{n=T-m}^{T-1} 1_F\circ\theta_n. \tag{2}$$

The number of occurrences of j_0 before T is the r.v. N defined by

$$\sum_{n=0}^{T-1} 1_{(X_n=j_0)}$$

and, according to [6, Theorem 3.5], the pre-T occupation measure identity for the chain starting at i is

$$E_i\left(\sum_{n=0}^{T-1} Y\circ\theta_n\right)=\sum_{j\in J} E_jYE_i\left(\sum_{n=0}^{T-1} 1_{(X_n=j)}\right) \tag{3}$$

where Y is any nonnegative r.v. on $(\Omega,\mathscr{F})$.

With $\delta(j_0,\ldots,j_k;i_0,\ldots,i_k)$ signifying the product for $s=0,\ldots,k$ of the Kronecker deltas with indices j_s,i_s, we prove

Theorem 2.1. *For every initial point i*

$$E_iN=\sum_{k=1}^{m} P(j_0,j_1)^{-1}\cdots P(j_{k-1},j_k)^{-1}\delta(j_0,\ldots,j_k;j_{m-k},\ldots,j_m)$$

Proof. We evaluate the expectation of the left-hand side of (2) by using (3) with $Y=1_F$ and the equation (1). Hence we obtain

$$E_i\left(\sum_{n=0}^{T-1} 1_F\circ\theta_n\right)=P(j_0,j_1)\cdots P(j_{m-1},j_m)E_iN. \tag{4}$$

We now extend the chain X backward in time so that it is stationary and compute the expectation of the right-hand side of (2) using the strong Markov property as follows:

$$E_i\left(\sum_{h=T-m}^{T-1} 1_F\circ\theta_h\right)=\sum_{h=0}^{m-1} E_i(1_F\circ\theta_{T+h-m})$$

$$=\sum_{h=0}^{m-1} E_i(E(1_F\circ\theta_{T+h-m}\,|\,X_{T-m},\ldots,X_T))$$

$$=\sum_{h=0}^{m-1} E_i(E(1_F\circ\theta_{h-m}\,|\,X_{-m}=j_0,\ldots,X_0=j_m))$$

$$=\sum_{h=0}^{m-1} E(1_F\circ\theta_h\,|\,X_0=j_0,\ldots,X_m=j_m)$$

$$=\sum_{h=0}^{m-1} P(X_h=j_0,\ldots,X_{h+m}=j_m\,|\,X_0=j_0,\ldots,X_m=j_m)$$

$$=\sum_{h=0}^{m-1} P(j_{m-h},j_{m-h+1})\cdots P(j_{m-1},j_m)\delta(j_0,\ldots,j_{m-h};j_h,\ldots,j_m)$$

$$=\sum_{k=1}^{m} P(j_k,j_{k+1})\cdots P(j_{m-1},j_m)\delta(j_0,\ldots,j_k;j_{m-k},\ldots,j_m).$$

The proof is completed by combining this last result with (2) and (4).

3. The main result

Let $T_1, T_2,\ldots$ denote the successive hitting times of j_0 by X. We now prove

Theorem 3.1. *For every initial point i*

$$E_iT=E_iT_1-E_{j_m}T_1+E_{j_0}T_1(\alpha+\delta(j_0;j_m)-\delta(j_0;i))$$

where α is the value of E_iN given by Theorem 2.1.

Proof. Define N_* by

$$N+1_{(X_T=j_0)}-1_{(X_0=j_0)}$$

so Theorem 2.1 takes the form

$$E_iN_*=\alpha+\delta(j_0;j_m)-\delta(j_0;i). \tag{5}$$

Since the events $(N_*>n)$ and $(T_{n+1}\leq T)$ are identical, it follows that N_* is a stopping time for the sequence $T_2-T_1, T_3-T_2,\ldots$ of independent r.v. with common expectation $E_{j_0}T_1$. Therefore Wald's equation gives

$$E_iT_{N_*+1} = E_iT_1 + E_i \sum_{n=1}^{N_*} (T_{n+1} - T_n) = E_iT_1 + E_iN_*E_{j_0}T_1. \tag{6}$$

Let $\mathscr{F}_T$ be the usual σ-field associated with T. Since

$$T_{N_*+1} - T = T_1 \circ \theta_T,$$

the strong Markov property shows that

$$E_i(T_{N_*+1} - T) = E_i(E(T_1 \circ \theta_T \mid \mathscr{F}_T)) = E_i(E_{X_T}T_1) = E_{j_m}T_1.$$

The theorem now follows by combining this last result with (5) and (6).

The ergodicity and Theorem 3.1 yield

Corollary 3.2. *Let $P(\cdot)$ be the stationary distribution, then*

$$E_{j_m}T = \alpha P(j_0)^{-1}.$$

4. Conclusion

Denoting by E the expectation operator when the initial distribution is the stationary distribution $P(\cdot)$, it follows from 3.1 that

$$ET = \sum_{i \in J} P(i)E_iT = ET_1 - E_{j_m}T_1 + E_{j_0}T_1(\alpha + \delta(j_0; j_m) - P(j_0)). \tag{7}$$

We will now recover Li's result in two simple versions; namely with or without an initial state i prescribed.

In the first situation Li's i.i.d. random variables $Z_1, Z_2, \ldots$ must be the r.v. $X_1, X_2, \ldots$ of our chain X, which now has the transition matrix of $Z_1, Z_2, \ldots$ and initial point i. Since with any initial distribution the expectation of T_1 is $P(j_0)^{-1}$ and since now

$$\alpha = P(j_0)\left(\sum_{k=0}^{m} P(j_0)^{-1} \cdots P(j_k)^{-1}\delta(j_0, \ldots, j_k; j_{m-k}, \ldots, j_m) \right) - \delta(j_0; j_m),$$

Lemma 2.4 in [5] follows from Theorem 3.1.

When Li does not assume any initial state, the r.v. $Z_1, Z_2, \ldots$ must be our r.v. $X_0, X_1, \ldots$, only now the initial distribution is the stationary and common distribution $P(\cdot)$. Moreover, when referred to $Z_1, Z_2, \ldots$, the waiting time for the sequence S to occur is $T+1$, because it is T when referred to $X_0, X_1, \ldots$. Now using (7) it is easy to see that

$$E(T+1) = \sum_{k=0}^{m} P(j_0)^{-1} \cdots P(j_k)^{-1}\delta(j_0, \ldots, j_k; j_{m-k}, \ldots, j_m)$$

in agreement with Lemma 2.4 in [5].

Acknowledgment

This paper was written while the author was visiting the Department of Statistics of the University of California at Berkeley. The author wishes to thank Professor D. Aldous for his helpful comments.

References

[1] G. Blom, On the mean number of random digits until a given sequence occurs, J. Appl. Prob. 19 (1982) 136–143.
[2] G. Blom and D. Thorburn, How many random digits are required until given sequences are obtained?, J. Appl. Prob. 19 (1982) 518–531.
[3] H.U. Gerber and S.-Y.R. Li, The occurrence of sequence patterns in repeated experiments and hitting times in a Markov chain, Stochastic Process. Appl. 11 (1981) 101–108.
[4] I.I. Glick, A note on runs, Z. Angew. Math. Mech. 61 (1981) 119–120
[5] S.-Y.R. Li, A martingale approach to the study of occurrence of sequence patterns in repeated experiments, Ann. Probability 8 (1980) 1171–1176.
[6] J.W. Pitman, Occupation measure for Markov chains, Adv. App. Prob. 9 (1977) 69–86.